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Iterated logarithm approximations to the distribution of the largest prime divisor

Arie Leizarowitz
Department of Mathematics
Technion, Haifa 32000
Israel

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Abstract

The paper is concerned with estimating the number of integers smaller than x whose largest prime divisor is smaller than y, denoted $\psi(x, y)$. Much of the related literature is concerned with approximating $\psi(x, y)$ by Dickman's function $\rho(u)$, where $u = \ln x / \ln y$. A typical such result is that

$$\psi(x, y) = x\rho(u)(1 + o(1)) \tag{1}$$

in a certain domain of the parameters x and y.

In this paper a different type of approximation of $\psi(x, y)$, using iterated logarithms of x and y, is presented. We establish that

$$\ln\left(\frac{\psi}{x}\right) = -u[\ln^{(2)} x - \ln^{(2)} y + \ln^{(3)} x - \ln^{(3)} y + \ln^{(4)} x - a] \tag{2}$$

where $\underline{a} < a < \bar{a}$ for some constants $\underline{a}$ and $\bar{a}$ (denoting by $\ln^{(k)} x = \ln \dots \ln x$ the k-fold iterated logarithm). The approximation (2) holds in a domain which is complementary to the one on which the approximation (1) is known to be valid. One consequence of (2) is an asymptotic expression for Dickman's function, which is of the form $\ln \rho(u) = -u[\ln u + \ln^{(2)} u](1 + o(1))$, improving known asymptotic approximations of this type. We employ (2) to establish a version of Bertrand's Conjecture, indicating how this method may be used to sharpen the result.

1 Introduction

A point $\mathbf{z}$ in R^m is *a lattice point* if $\mathbf{z} = (z_1, ..., z_m)$ where each z_j is an integer. Consider the number of lattice points included in the simplex $S(a_1, ..., a_m)$, where

$$S(a_1, ..., a_m) = \left\{ \mathbf{z} : \sum_{j=1}^{m} \frac{z_j}{a_j} \leq 1,\ z_j \geq 0, 1 \leq j \leq m \right\},$$

and $a_j,\ j = 1, 2, ..., m$, are positive real numbers. Denote this number by $\lambda(a_1, ..., a_m)$, or $\lambda(S)$.

We need estimates of $\lambda(S)$ as a tool in studying the following problem: *Let x and y be two positive real numbers, and we are interested in the number of integers $2 \leq k \leq x$ such that the largest prime divisor of k does not exceed y, denoted $\psi(x, y)$.*

Denote by $\{p_j\}_{j=1}^{\infty}$ the increasing sequence of the primes, and let m be such that

$$p_m < y \leq p_{m+1}.$$

Then by the Prime Numbers Theorem

$$m \approx \frac{y}{\ln y} \tag{1.1}$$

in the sense that the ratio between the two sides of (1.1) tends to 1 as $y \to \infty$. We are thus interested in the integers $k \leq x$ which are of the form

$$k = \prod_{j=1}^{m} p_j^{t_j},\ t_j \text{ are nonnegative integers.} \tag{1.2}$$

Equivalently, we are interested in integers k as in (1.2) for which

$$\sum_{j=1}^{m} (\ln p_j) t_j \leq \ln x \tag{1.3}$$

holds. Thus to approximate $\psi(x, y)$ we estimate the expression

$$\lambda \left(\frac{\ln x}{\ln p_1}, ..., \frac{\ln x}{\ln p_m} \right).$$

There has appeared quite extensive literature on the subject of integers without large prime divisors since the 30' of the previous century. See e.g.

Dickman [3], Erdősh [4, 5, 6], Erdősh and Schinzel [7], Fouvry and Tenenbaum [8], Friedlander [9, 10, 11, 12], Granville [13, 14, 15, 16], Hazlewood [17], Hildebrand [18, 19, 20, 21, 22], Hildebrand and Tenenbaum [23, 24], Pomerance [28, 29], Ramachandra [30, 31, 32], Rankin [33], Tenenbaum [38, 39], Vershik [41], Xuan [42, 43], and the survey paper by Hildebrand and Tenenbaum [25]. More recent related work is presented in de la Bretèche and Tenenbaum [1], Hunter [26], Scourfield [37], Song [34], Suzuki [35, 36] and Tenenbaum [40],

Dickman [3] has established that for every fixed $u > 1$ the limit

$$\lim_{x\to\infty} \frac{\psi(x, x^{1/u})}{x} = \rho(u) \tag{1.4}$$

exists, where $\rho(u)$ is the unique continuous solution of

$$u\rho'(u) = -\rho(u-1), u \geq 1$$

satisfying

$$\rho(u) = 1 \text{ for } 0 \leq u \leq 1.$$

It turns out that ρ satisfies the asymptotic relation

$$\ln \rho(u) = -(1 + o(1))u \ln u. \tag{1.5}$$

Concerning $\psi(x, y)$ we obtained the following result, which is implied by our main results, Theorems 7.1 and 7.2. It deals with situations where

$$\ln x << y << x,$$

in a sense expressed precisely Theorem 1.1. We employ the notation

$$\ln^{(k)} x = \ln \cdots \ln x \tag{1.6}$$

for the kth iterated logarithm, where the logarithm function appears k times in the right hand side of (1.6) and x is sufficiently large. Namely,

$$\ln^{(1)} x = \ln x,\ \ln^{(k+1)} x = \ln(\ln^{(k)} x),\ k \geq 1.$$

Theorem 1.1 *(i) Consider pairs (x, y) such that*

$$\exp{(\ln y)^{1-\theta}} < \ln x < \sqrt{y} \tag{1.7}$$

for some $0 < \theta < 1$. *Denoting*

$$u = \frac{\ln x}{\ln y}, \tag{1.8}$$

there exist constants $\underline{a} > 0$ *and* $y_0 > 1$ *such that*

$$\ln\left(\frac{\psi(x,y)}{x}\right) > -u[\ln^{(2)} x - \ln^{(2)} y + \ln^{(3)} x - \ln^{(3)} y + \ln^{(4)} x - \underline{a}] \tag{1.9}$$

for every $y > y_0$.
(ii) Consider pairs (x, y) *such that*

$$(\ln y)^\nu < \ln x < y^\beta \tag{1.10}$$

for some $\nu > 2$ *and some* $0 < \beta < 1/2$. *Then there exist constants* $\overline{a} > a^\star$ *and* $y_0 > 1$ *such that*

$$\ln\left(\frac{\psi(x,y)}{x}\right) < -u[\ln^{(2)} x - \ln^{(2)} y + \ln^{(3)} x - \ln^{(3)} y + \ln^{(4)} x - \overline{a}] \tag{1.11}$$

for every $y > y_0$.

We use the estimates of the iterated logarithms of x and y described in Proposition 7.5 and the inequalities (1.9) and (1.11) to obtain the following strengthening of (1.5).

Corollary 1.2 *Consider pairs* (x, y) *such that* (1.7) *holds, and let* u *be as in* (1.8). *Then*

$$\ln \rho(u) = -u(\ln u + \ln^{(2)} u)(1 + o(1)), \tag{1.12}$$

where the term $o(1)$ *is of order* $O\left(\frac{\ln^{(3)} u}{\ln^{(2)} u}\right)$.

Assuming validity of the conjectured expressions (7.6) and (7.7) in Remark 7.3 yields that the $o(1)$ term in (1.12) is of order $O\left(\frac{\ln^{(k)} u}{\ln^{(2)} u}\right)$ for any $k \geq 3$.

Another application of (1.9) and (1.11) is to Bertrand's Conjecture, expressed in Corollary 7.6, establishing that for every $\gamma > 3/2$ there exists y_0 such that

$$y < p < \gamma y$$

for some prime p, if $y > y_0$. There exist stronger results concerning Bertrand's Conjecture (see e.g. [27]), and we present Corollary 7.6 to demonstrate the efficiency of our main results Theorems 7.1 and 7.2 as a tool in studying certain interesting problems.

A uniform version of Dickman's result (1.4) was established by de Bruijn [2]. Using u in (1.8) he has proved that

$$\psi(x, y) = x\rho(u) \left\{1 + O\left(\frac{\ln u}{\ln y}\right)\right\} \tag{1.13}$$

holds uniformly in the domain

$$2 \leq u \leq (\ln y)^{3/5-\epsilon},\ y \geq 2.$$

This asymptotic relation was extended by Hildebrand [20] who proved that (1.13) holds uniformly in the domain

$$2 \leq u \leq \exp\{(\ln y)^{3/5-\epsilon}\},\ y \geq 2. \tag{1.14}$$

The upper limit of the domain of validity of (1.13) is related to the error term in the Prime Number Theorem. Actually Hildebrand established in [18] that Riemann Hypothesis is true if and only if (1.13) holds uniformly in the domain

$$2 \leq u \leq y^{1/2-\epsilon},\ y \geq 2, \tag{1.15}$$

for any fixed $\epsilon > 0$.

Remark 1.3 *Note that restricting to the domain* (1.15)*, the domain*

$$u < \exp(\ln y)^{3/5-\epsilon}$$

in (1.14) *is complementary to the domain*

$$\ln x > \exp{(\ln y)}^{1-\theta}$$

in (1.7) *for* $\theta > 2/5 + \epsilon$.

Remark 1.4 *The expressions* (1.9) *and* (1.11) *provide approximations of* $\ln(\psi(x, y)/x)$*, whose accuracy is expressed by* $(\overline{a} - \underline{a})u$*. To attain the same*

level of accuracy as in the approximations (1.13) *it is required that* $\overline{a} - \underline{a} = O(\ln u / \ln x)$*, or equivalently*

$$\overline{a} - \underline{a} = O\left(\frac{\ln \ln x}{\ln x}\right). \tag{1.16}$$

Moreover, it is easy to see from the proof of Corollary 7.6*, that* (1.16) *implies the following result for Bertrand's Problem: For every* $\epsilon > 0$ *there exists a* y_0 *such that if* $y > y_0$ *then*

$$y < p < y + y^{1/2+\epsilon} \text{ for some prime } p.$$

We conclude with a result that covers the following range of (x, y)

$$\frac{1}{2} \ln x < \ln y < \ln x, \tag{1.17}$$

which is different from the ranges indicated in Theorem 1.1.

Theorem 1.5 *Consider the set* E *of integers* $1 \leq k \leq x$ *for which all the prime divisors are smaller than* $\sqrt{x}$*. (In our notations* $\#(E) = \psi(x, \sqrt{x})$*.) If* (x, y) *satisfies* (1.17) *then*

$$\psi(x, y) \geq \#(E) > \alpha x \text{ for some constant } \alpha > 0 \text{ and every } x > 1. \tag{1.18}$$

Actually, for sufficiently large x *we may take* $\alpha = \ln(e/2)$ *in* (1.18)*.*

The proof is relegated to the appendix.

The paper is organized as follows. In the next section we describe a convenient setting for the study of lower and upper bounds of $\psi(x, y)$. In section 3 we introduce a family of auxiliary problems in which our problem can be imbedded. In section 4 we introduce our iterations method, which is the main technical tool developed in this paper. In sections 5 and 6 we establish lower and upper bounds for the auxiliary problems defined in section 3, respectively. Our main results are presented in section 7. In the appendix we establish Theorem 1.5 and Proposition 6.2.

2 The reduced order simplex

In this section we relate with the high dimensional simplex (1.3) a simplex of smaller order. We will study certain properties of this simplex, which will be used in the next sections as tools used to establish tight lower and upper bounds for the number of solutions of (1.3).

We divide the integers interval $(1, y)$ into subintervals

$$J_i = \left(\frac{y}{e^i}, \frac{y}{e^{i-1}}\right), i = 1, 2, ..., r, \tag{2.1}$$

where

$$r = [\ln y] \text{ if } \ln y < [\ln y] + \ln 2 \tag{2.2}$$

and

$$r = [\ln y] + 1 \text{ if } \ln y > [\ln y] + \ln 2. \tag{2.3}$$

For simplicity of notations we henceforth consider only case (2.2), and comment that the discussion and main results in case (2.3) are the same. (In Remark 2.1 we will indicate where the difference between (2.2) and (2.3) plays a role.)

We have for primes $p_j \in J_i$ the relations

$$\ln y - i < \ln p_j < \ln y - i + 1, \tag{2.4}$$

and regarding (1.3) this implies

$$(\ln y - i)z_i < \sum_{p_j \in J_i} (\ln p_j)t_j < (\ln y - i + 1)z_i, \tag{2.5}$$

where we denote

$$z_i = \sum_{p_j \in J_i} t_j, \ 1 \le i \le r. \tag{2.6}$$

Clearly $(z_1, ..., z_r)$ is a nonnegative lattice point in R^r.

Remark 2.1 *The cases* (2.2) *and* (2.3) *differ only when considering* $i = r$ *in the left inequality of* (2.4).

If $\{t_j\}_{j=1}^m$ is a solution of (1.3), then in view of (2.5) this implies

$$\sum_{i=1}^{r} (\ln y - i)z_i < \ln x. \tag{2.7}$$

Therefore the number of solutions $\{t_j\}_{j=1}^m$ of (1.3) is smaller than the number of solutions $\{t_j\}_{j=1}^m$ of (2.7). (We say that $\{t_j\}_{j=1}^m$ is a solution of (2.7) if (2.6) and (2.7) are satisfied.) Similarly, if $\{t_j\}_{j=1}^m$ is a solution of

$$\sum_{i=1}^{r}(\ln y - i + 1)z_i < \ln x, \tag{2.8}$$

then in view of (2.5) it is also a solution of (1.3), implying that the number of solutions $\{t_j\}_{j=1}^m$ of (1.3) is larger than the number of solutions $\{t_j\}_{j=1}^m$ of (2.8). These considerations are the basis of our computation of upper and lower bounds for $\psi(x,y)$.

For a prescribed lattice point $(z_1,...,z_r)$ which satisfies (2.7) we are interested in the number of lattice points $\{t_j\}_{j=1}^m$ in R^m for which (2.6) holds for every $i = 1,2,...,r$. Let m_i denote the size of the set $\{j : p_j \in J_i\}$:

$$m_i = \#\left\{p_j \in \left(\frac{y}{e^i}, \frac{y}{e^{i-1}}\right)\right\},$$

and if $m_i >> 1$, then by the Prime Numbers Theorem

$$m_i \approx \frac{(e-1)y}{(\ln y - i)e^i}, \tag{2.9}$$

and we have the inequality

$$m_i > \frac{y}{e^i(\ln y - i)}. \tag{2.10}$$

We denote by $f(k,m)$ the number of different ways in which k can be written as a sum of m nonnegative integers, and clearly

$$f(k,m) = \left(\begin{array}{c} k+m-1 \\ k \end{array}\right) = \frac{m(m+1)\cdots(m+k-1)}{k!}. \tag{2.11}$$

Then the number of lattice points $\{t_j\}_{j=1}^m$ that satisfy (2.6) for every $1 \le i \le r$ is

$$K(z_1,...z_r) = \prod_{i=1}^{r} f(z_i, m_i). \tag{2.12}$$

We Denote by $\overline{\psi}(x,y)$ and $\underline{\psi}(x,y)$ the number of solutions of (2.7) and (2.8) respectively, and it follows that $\psi(x,y)$ is bounded from above by $\overline{\psi}(x,y)$

and from below by $\underline{\psi}(x, y)$. Using the expression $K(z_1, ..., z_r)$ in (2.12) we consider sums of the form

$$M(F) = \sum_{\mathbf{z} \in F} K(z_1, ..., z_r), \tag{2.13}$$

where the summation runs over all the lattice points $\mathbf{z} = \{z_1, ..., z_r\}$ which belong to some set F in R^r. Thus when F in (2.13) is the set of points belonging to the simplex (2.7), denoted F_1, then by (2.11) and (2.12) we have

$$\overline{\psi}(x, y) = \sum_{\{z_i\} \in F_1} \prod_{i=1}^{r} \frac{m_i^{z_i}}{z_i!} \left(1 + \frac{1}{m_i}\right) \cdots \left(1 + \frac{z_i - 1}{m_i}\right). \tag{2.14}$$

Similarly we obtain the following lower bound for ψ

$$\underline{\psi}(x, y) = \sum_{\{z_i\} \in F_2} \prod_{i=1}^{r} \frac{m_i^{z_i}}{z_i!}, \tag{2.15}$$

where F_2 is the set of all the lattice points in the simplex (2.8).

We next consider the product

$$P_i = \prod_{k=1}^{z_i - 1} \left(1 + \frac{k}{m_i}\right)$$

that appears in the right hand side of (2.14), and in view of the inequality $\ln(1 + t) < t$ for $t > 0$ we obtain $\ln P_i < z_i^2 / 2m_i$, hence

$$P_i < e^{z_i^2 / 2m_i}. \tag{2.16}$$

When dealing with a lower bound we will ignore the term $\prod_{i=1}^{r} P_i$ in the right hand side of (2.14), and we will focus on computing a lower bound to expressions of the form

$$Z(F) = \sum_{\{z_i\} \in F} \prod_{i=1}^{r} \frac{m_i^{z_i}}{z_i!} \tag{2.17}$$

for certain sets F. We will then describe the modifications required to obtain an upper bound by taking into consideration the terms P_i in (2.14).

3 A family of auxiliary problems

It will be convenient to study our main problem, of estimating sums of the form (2.13), by using slightly different notations. In this section we define a collection of problems, parameterized by two real variables, such that for certain values of the parameters the auxiliary problem coincides with the main problem. Thus for a positive number $c > 1$, let $r = [c]$ and consider the inequality

$$cz_0 + (c-1)z_1 + (c-2)z_2 + \cdots + (c-r+1)z_{r-1} < M \tag{3.1}$$

for some positive number $M > 1$, where $\mathbf{z} = \{z_i\}_{i=0}^{r-1}$ is a nonnegative lattice point in R^r (compare with (2.8)). We associate with c the r bases

$$m_i = \frac{(e-1)e^{c-i}}{c-i},\ 0 \leq i \leq r-1 \tag{3.2}$$

(compare with (2.9) in case that $c = \ln n$). In view of (2.15) we address the problem of computing the sum

$$F(c, M) = \sum_{\mathbf{z}} \prod_{i=0}^{r-1} \frac{m_i^{z_i}}{z_i!}, \tag{3.3}$$

where $\mathbf{z} = (z_0, ..., z_{r-1})$ runs over all the nonnegative lattice points which satisfy (3.1); we call this *Problem* $P_{c,M}$ for the r variables z_0,...,z_{r-1}.

Remark 3.1 *There is a close relation between the value of Problem* $P_{c,M}$ *and* $\psi(x, y)$ *for*

$$c = \ln y \text{ and } M = \ln x. \tag{3.4}$$

Thus the value of $P_{c,M}$ *yields a lower bound for* $\psi(x, y)$. *We also note that if* $c \geq M$ *(namely* $y \geq x$*) and* x *is an integer, then*

$$\psi(x, y) = x = e^M. \tag{3.5}$$

To establish an upper bound for $\psi(x, y)$ we will estimate a sum of the type (2.13), which is associated with the simplex

$$(c-1)z_1 + (c-2)z_2 + \cdots + (c-r)z_r < M \tag{3.6}$$

(compare with (2.7)). This sum is smaller than the corresponding sum that is associated with the simplex

$$cz_0 + (c-1)z_1 + (c-2)z_2 + \cdots + (c-r)z_r < M, \tag{3.7}$$

which we denote by $G_0(c, M)$. Thus to obtain an upper bound for $G_0(c, M)$ we consider a sum similar to the one in (3.3), where we take into consideration the terms P_i in (2.16). We then address the problem of computing the sum

$$G(c, M) = \sum_{\mathbf{z}} \prod_{i=0}^{r} \frac{m_i^{z_i} e^{z_i^2/m_i}}{z_i!}, \tag{3.8}$$

where $\mathbf{z} = (z_0, z_1, ..., z_r)$ runs over all the nonnegative lattice points which satisfy (3.7); we call this *Problem* $Q_{c,M}$ for the $r+1$ variables z_0,z_1,...,z_r.

Remark 3.2 *We use the simplex* (3.7) *rather than the simplex* (3.6)*, which is more directly related to* (2.7)*, in order to avoid repetition of computations for the lower and upper bounds. Thus a substantial part of the computations for* (3.1) *and* (3.7) *will be unified.*

We claim that for a fixed value of z_0, Problem $P_{c,M}$ reduces to Problem $P_{c-1,M-cz_0}$ for the $r-1$ variables z_1,...,z_{r-1}. To justify this statement we have to check that the $r-1$ bases m_1,...,m_{r-1} in (3.2) are indeed the bases associated with Problem $P_{c-1,M-cz_0}$, which is easily verified.

The possible values for the variable z_0 in (3.1) are the integers z satisfying

$$0 \leq z \leq \frac{M}{c},$$

and it follows from (3.3) that

$$F(c, M) = \sum_{z=0}^{[M/c]} F(c-1, M-cz) \frac{m_0^z}{z!}. \tag{3.9}$$

In the subsequent discussion we will consider situations where $F(\cdot, \cdot)$ satisfies inequalities of the form

$$F(c, M) \geq B e^{M\left(1 - \frac{\ln M}{c+1} + \frac{\gamma}{c+1}\right)} \tag{3.10}$$

for some constant $0 < B \le 1$. In terms of the original parameters we are actually interested in inequalities of the form

$$\psi(x,y) \ge Bx^{\left(1-\frac{\ln\ln x}{\ln y+1}+\frac{\gamma}{\ln y+1}\right)}, \tag{3.11}$$

where (x,y) and (c,M) are related as in (3.4).

Remark 3.3 *We note that M/c is the parameter u in* (1.8)*, which appeared, e.g., in* (1.13)*,* (1.14) *and* (1.15)*. It follows from* (1.18) *in Theorem* 1.5 *that for a fixed γ, inequality* (3.11) *holds whenever $M/c < 2$. Indeed, for $M = \ln x$ and $c = \ln y$ the condition $M/c < 2$ translates to $y > \sqrt{x}$, implying $\psi(x,y) > \alpha x$ by* (1.18)*. But the inequality*

$$\alpha x > x^{1-\frac{\ln\ln x}{\ln y+1}+\frac{\gamma}{\ln y+1}}$$

is equivalent to

$$\frac{\ln x}{\ln y+1}(\ln\ln x - \gamma) > -\ln\alpha,$$

and this holds for every $x > x_0$, for some x_0, since $y < x$. For $x \le x_0$, however, (3.11) *holds for some $B(\gamma)$, since in this case we have a bounded set of pairs (x,y). Therefore, when trying to establish an inequality of the type* (3.10)*, we may assume that*

$$\frac{M}{c} \ge 2, \tag{3.12}$$

since for $M/c < 2$ inequality (3.11) *is already established.*

4 The iterations method

The discussion in this section is fundamental to our analysis. We develop the iterations method which will be employed in the subsequent sections to establish lower and upper bounds for ψ.

Assume that for a certain $\gamma > 0$ and some $0 < B < 1$, inequality (3.10) holds for any pair (c,M) which verifies

$$c \le \kappa_0, \tag{4.1}$$

for a certain κ_0. We consider then pairs (c, M) that satisfy

$$\kappa_0 < c \leq \kappa_0 + 1, \tag{4.2}$$

and our goal is to establish the inequality (3.10) for such pairs as well. Once this is achieved we will iterate the argument to obtain a lower bound for all pairs in a certain domain.

Intending to employ (3.9) to establish a lower bound to $F(c, M)$, and assuming that (3.10) holds whenever (4.1) is satisfied, we will estimate from below the expressions

$$F(c-1, M-cz)\frac{m_0^z}{z!} \tag{4.3}$$

which appear in (3.9), and this for integers $0 \leq z \leq M/c$. By (4.2) $c-1 \leq \kappa_0$, and we may use (3.10) for the pair $(c-1, M-cz)$, obtaining

$$F(c-1, M-cz) \geq Be^A, \tag{4.4}$$

where

$$A = (M-cz) - \frac{1}{c}(M-cz)\ln(M-cz) + \frac{(M-cz)\gamma}{c}. \tag{4.5}$$

Moreover, the inequality

$$\frac{m_0^z}{z!} > e^E, \tag{4.6}$$

holds, where we denote

$$E = (z\ln m_0 - z\ln z + z) - \left(\frac{1}{2}\ln z + \frac{1}{2}\ln\pi + \frac{3}{2}\ln 2\right), \tag{4.7}$$

where we used Stirling's formula

$$St(z) = \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \tag{4.8}$$

to estimate

$$z! < 2St(z) \text{ for every } z \geq 1. \tag{4.9}$$

In (4.7), a term $(-\ln 2)$ arises from the factor 2 in (4.9), and the term

$$-\frac{1}{2}(\ln z + \ln\pi + \ln 2) \tag{4.10}$$

is due to the logarithm of $\sqrt{2\pi z}$ in (4.8). To avoid the disturbing term (4.10) in (4.7) we note that

$$\frac{1}{2}\ln z + \frac{1}{2}\ln \pi + \frac{3}{2}\ln 2 < hz \tag{4.11}$$

where $h > 0$ may be chosen arbitrarily small provided that z is sufficiently large. It follows that

$$z - \left(\frac{1}{2}\ln z + \frac{1}{2}\ln \pi + \frac{3}{2}\ln 2\right) > bz \tag{4.12}$$

where

$$b = 1 - h \tag{4.13}$$

may be chosen arbitrarily close to 1 provided that z is large enough, and we thus obtain

$$E > (z \ln m_0 - z\ln z + bz) \tag{4.14}$$

for sufficiently large values of z.

It follows from $m_0 = (e-1)e^c/c$ that

$$z \ln m_0 = cz - z\ln c + z\ln(e-1).$$

Using the last equation in (4.14) and recalling (4.5) yield that

$$A + E > H(z), \tag{4.15}$$

denoting

$$H(z) = M\left(1 + \frac{\gamma}{c}\right) + (a-\gamma)z - \frac{M}{c}\ln c - z\ln z - \left(\frac{M}{c} - z\right)\ln\left(\frac{M}{c} - z\right) \tag{4.16}$$

and

$$a = b + \ln(e-1). \tag{4.17}$$

Thus a is smaller and arbitrarily close to $a^\star$, which is defined by

$$a^\star = 1 + \ln(e-1). \tag{4.18}$$

It follows from (4.4), (4.6) and (4.15) that

$$F(c-1, M-cz)\frac{m_0^z}{z!} > Be^{H(z)}, \tag{4.19}$$

and to obtain a lower bound for the sum in (3.9) we will estimate the maximal value of $H(z)$, $0 \le z \le [M/c]$, where z is an integer.

Remark 4.1 *We will compute a maximizer z_0 of $H(\cdot)$ defined on the real interval $[0, [M/c]]$, and in general z_0 is not an integer. Let z_1 be the integer*

$$z_1 = z_0 + \theta \text{ for some } 0 \leq \theta < 1,$$

and then

$$H(z_1) = H(z_0) + \frac{1}{2} H''(\zeta)\theta^2$$

for some $z_0 < \zeta < z_1$. But

$$H''(\zeta) = \frac{-M/c}{\zeta(M/c - \zeta)},$$

and it follows from $\zeta \geq 1$ that

$$|H''(\zeta)| \leq \frac{M/c}{M/c - 1} < 2$$

(since $M/c > 2$), and we obtain

$$H(z_1) > H(z_0) - \theta^2. \tag{4.20}$$

Similarly, for the integer $z_2 = z_0 - (1 - \theta)$ we have

$$H(z_2) > H(z_0) - (1 - \theta)^2. \tag{4.21}$$

It follows from (4.19), (4.20) and (4.21) that

$$\sum_{z=0}^{[M/c]} F(c-1, M - cz)\frac{m_0^z}{z!} > B\left(e^{H(z_1)} + e^{H(z_2)}\right) > Be^{H(z_0)} \tag{4.22}$$

since

$$\min_{0 \leq \theta \leq 1} \left\{ e^{-\theta^2} + e^{-(1-\theta)^2} \right\} > 1.$$

Therefore we may use the maximal value of $H(z)$ over the whole real interval $0 \leq z \leq M/c$.

We have the following basic result.

Proposition 4.2 *Let $H(z)$ be as in (4.16). Then*

$$\max\left\{H(z) : 0 \le z \le \frac{M}{c}\right\} = M\left(1 - \frac{\ln M}{c} + \frac{\gamma + f(\gamma)}{c}\right), \tag{4.23}$$

where

$$f(\gamma) = \ln(1 + e^{a-\gamma}). \tag{4.24}$$

Proof: Denoting

$$u = \frac{M}{c} \text{ and } z = ut$$

it follows that

$$\begin{aligned} &\max_z\{(a-\gamma)z - z\ln z - (u-z)\ln(u-z)\} = \\ &-u\ln u + u\max_{0\le t\le 1}\{(a-\gamma)t - t\ln t - (1-t)\ln(1-t)\}. \end{aligned} \tag{4.25}$$

We denote

$$\varphi(t) = (a-\gamma)t - t\ln t - (1-t)\ln(1-t), \tag{4.26}$$

and it follows that the maximizer t_0 of φ satisfies

$$(a-\gamma) - \ln t_0 + \ln(1-t_0) = 0.$$

We conclude that

$$t_0(\gamma) = \frac{1}{1+e^{\gamma-a}}, \tag{4.27}$$

and the maximal value of $\varphi(\cdot)$ is given by

$$(a-\gamma)t_0 + \ln(1+e^{\gamma-a}) - (1-t_0)(\gamma-a),$$

which yields

$$\max\{\varphi(t) : 0 \le t \le 1\} = \ln(1+e^{a-\gamma}). \tag{4.28}$$

We thus conclude from (4.25) and (4.28) that

$$\begin{aligned} &\max_{0\le z\le u}\left\{z(a-\gamma) - z\ln z - \left(\frac{M}{c} - z\right)\ln\left(\frac{M}{c} - z\right)\right\} = \\ &\qquad -u\ln u + u\ln(1+e^{a-\gamma}). \end{aligned} \tag{4.29}$$

It follows from (4.16) and (4.29) that (4.23) is satisfied, where $f(\gamma)$ in (4.24) is the maximum in (4.28). The proof of the proposition is complete. □

Proposition 4.3 *Assume that*

$$F(c', M) \geq Be^{M\left(1-\frac{\ln M}{c'+1}+\frac{\gamma}{c'+1}\right)}$$

holds for every $c' \leq c-1$*, for some* $c > 1$*. Then*

$$F(c, M) \geq B \exp\left\{M\left(1 - \frac{\ln M}{c} + \frac{\gamma + f(\gamma)}{c}\right)\right\}. \tag{4.30}$$

Proof: Equation (4.30) follows from (3.9), (4.22) and (4.23). $\square$

For the induction argument we need that (3.10) would hold for some initial value of c, say for $c = \kappa$ for some $\kappa > 1$. This is the content of the following result.

Proposition 4.4 *For a prescribed* $\gamma > 0$ *the inequality*

$$F(\kappa, M) \geq B(\kappa, \gamma)e^{M\left(1-\frac{\ln M}{\kappa+1}+\frac{\gamma}{\kappa+1}\right)} \tag{4.31}$$

holds for every $M \geq 0$*, where*

$$B(\kappa, \gamma) = e^{-e^{\kappa+\gamma}}. \tag{4.32}$$

Proof: The maximal value of

$$M \mapsto M\left(1 - \frac{\ln M}{\kappa+1} + \frac{\gamma}{\kappa+1}\right)$$

is $\dfrac{e^{\kappa+\gamma}}{\kappa+1}$, and it is attained at $M_0 = e^{\kappa+\gamma}$. Since $B(\kappa, \gamma)$ in (4.32) satisfies

$$B(\kappa, \gamma)e^{\frac{e^{\kappa+\gamma}}{\kappa+1}} < 1,$$

and since $F(c, M) \geq 1$, inequality (4.31) follows for every $M > 1$. $\square$.

It follows from Proposition 4.4 that if B in (3.10) is equal to $B(\kappa, \gamma)$ in (4.32), then (3.10) holds for any pair (c, M) such that $c \leq \kappa$.

5 A lower bound for Problem $P_{c,M}$

In this section we employ the results of the previous section to establish a lower bound for Problem $P_{c,M}$. We will construct a sequence

$$\{(c_j, M_j)\}_{j=0}^{l}$$

(where $c_{j+1} = c_j - 1$), for which (4.30) will be employed successively. The coefficient B will be chosen such that

$$F(c, M) \geq B \exp\left\{M\left(1 - \frac{\ln M}{c+1} + \frac{\gamma'}{c+1}\right)\right\} \tag{5.1}$$

will hold for the pair (c_l, M_l) for a certain $\gamma' = \gamma_l$, and consequently, employing (4.30), it will hold for each (c_j, M_j) with a corresponding $\gamma' = \gamma_j$. In particular it will hold for $(c, M) = (c_0, M_0)$ with a certain $\gamma' = \gamma_0$.

Recall that in deriving the estimate (4.30) we used a value

$$z_0 = ut_0$$

and that we associated with (c_0, M_0) a pair (c_1, M_1) such that $c_1 = c_0 - 1$, and

$$M_1 = M_0(1 - t_0). \tag{5.2}$$

Although this pair does not correspond to an integer z, it may be used in computing a lower bound for $F(c, M)$, as explained in Remark 4.1.

Let a be associated with z_0 as in (4.11), (4.13) and (4.17). Recalling (4.18) we have the following result:

Proposition 5.1 *For any prescribed $\epsilon > 0$ there exists a u_0 such that*

$$|a - a^\star| < \epsilon \textit{ if } u > u_0. \tag{5.3}$$

Concerning (4.30), we wish to estimate its right hand side as follows:

$$M\left(1 - \frac{\ln M}{c} + \frac{\gamma + f(\gamma)}{c}\right) > M\left(1 - \frac{\ln M}{c+1} + \frac{\gamma'}{c+1}\right) \tag{5.4}$$

for a certain γ'. Clearly the inequality (5.4) is equivalent to

$$\frac{\gamma + f(\gamma)}{c} > \frac{\ln M}{c(c+1)} + \frac{\gamma'}{c+1}. \tag{5.5}$$

For any $\beta > 0$ we denote

$$\mathcal{D}_\beta = \{(c, M) : 1 \leq M \leq e^{\beta c}\}, \tag{5.6}$$

and for a fixed $\alpha > 0$ and a pair (c, M) we denote

$$\gamma_{c,M} = a - \alpha + \ln c - \ln \ln M. \tag{5.7}$$

We assume the validity of (5.1) with $c - 1$ replacing c and with

$$\gamma' = \gamma_{c-1,M-cz}.$$

Namely we assume that

$$F(c-1, M') \geq B \exp\left\{M'\left(1 - \frac{\ln M'}{c} + \frac{\gamma_{c-1,M'}}{c}\right)\right\} \tag{5.8}$$

for every $1 \leq M' \leq M$. Using (5.7) in (5.8) yields

$$F(c-1, M') \geq B \exp\left\{M'\left(1 - \frac{\ln M'}{c} + \frac{a - \alpha + \ln(c-1) - \ln \ln M}{c}\right)\right\},$$

which we write in the form

$$F(c-1, M') \geq B \exp\left\{M'\left(1 - \frac{\ln M'}{c} + \frac{\gamma_0}{c}\right)\right\} \tag{5.9}$$

for every $1 \leq M' \leq M$, denoting

$$\gamma_0 = a - \alpha + \ln(c-1) - \ln \ln M. \tag{5.10}$$

For a pair (c, M) we consider the maximization over z of

$$F(c-1, M - cz)\frac{m_0^z}{z!}. \tag{5.11}$$

The fact that the parameter γ_0 in (5.9) is one and the same for all M' enables to employ the results of section 4. Thus the maximal value of (5.11) exceeds the maximal value which is obtained when we replace $F(c-1, M-cz)$ by the right hand side of (5.9), with $M' = M - cz$, namely the maximal value of

$$B \exp\left\{(M - cz)\left[1 - \frac{\ln(M - cz)}{c} + \frac{\gamma_0}{c}\right]\right\}\frac{m_0^z}{z!} \tag{5.12}$$

over $0 \le z \le M/c$. This latter maximum is attained at

$$M' = M(1 - t_0) \tag{5.13}$$

where

$$t_0 = \frac{1}{1 + e^{\gamma_0 - a}}. \tag{5.14}$$

We focus our attention on the domain $\mathcal{D}_{1/2}$ (recall (5.6)), and will next establish that if $(c, M) \in \mathcal{D}_{1/2}$ then also the resulting pair $(c-1, M')$ belongs to $\mathcal{D}_{1/2}$.

Proposition 5.2 *There exists an $\alpha_0 > 0$ and $c_0 > 0$ such that if α in* (5.7) *satisfies $\alpha > \alpha_0$ then for $c > c_0$*

$$(c, M) \in \mathcal{D}_{1/2} \Rightarrow (c-1, M') \in \mathcal{D}_{1/2}. \tag{5.15}$$

Proof: By (5.10)

$$e^{\gamma_0 - a} = e^{-\alpha} \frac{c-1}{\ln M}, \tag{5.16}$$

and since $(c, M) \in \mathcal{D}_{1/2}$, we have $(\ln M)/c \le 1/2$. We distinguish between the situation where $(\ln M)/c$ is close to $1/2$, and where $(\ln M)/c$ is smaller, say

$$\frac{\ln M}{c} < \mu \tag{5.17}$$

for some $0 < \mu < 1/2$. If (5.17) holds then for some c_0 we have

$$\frac{\ln M'}{c-1} < \frac{\ln M}{c-1} < \frac{1}{2}$$

for every $c > c_0$. If, however, (5.17) does not hold, so that

$$\frac{c}{\ln M} \le \frac{1}{\mu}, \tag{5.18}$$

then we obtain from (5.14) and (5.16) that $t_0 < 1$ is arbitrarily close to 1 provided that α is large enough. In particular we have that

$$-\ln(1 - t_0) > \frac{1}{2},$$

which implies $(\ln M')/(c-1) < 1/2$ in view of $\ln M' = \ln M + \ln(1 - t_0)$. The proof is complete. $\square$

Remark 5.3 *We consider pairs* $(c, M) \in \mathcal{D}_\beta$ *where we let* $\beta \to 0$. *It then follows from* (5.10) *and* (5.14) *that*

$$\ln(1 - t_0) < -ke^{a-\gamma_0} \tag{5.19}$$

for some $0 < k < 1$. *Actually* k *is arbitrarily close to* 1 *if* β *is sufficiently small, since then, by* (5.10), γ_0 *becomes arbitrarily large, using*

$$\ln c - \ln\ln M \approx \ln(1/\beta).$$

It follows from (5.10) *that*

$$e^{a-\gamma_0} = e^{\alpha}\frac{\ln M}{c-1}, \tag{5.20}$$

and employing

$$\frac{\ln M'}{c-1} = \frac{\ln M + \ln(1-t_0)}{c-1}$$

we conclude from (5.19) *and* (5.20) *that we have*

$$\frac{\ln M'}{c-1} < \frac{\ln M(c-1-ke^{\alpha})}{c(c-1)} < \frac{\ln M}{c} < \beta \tag{5.21}$$

for $\alpha > \alpha(\beta)$, *where* $\alpha(\beta) \to 0$ *as* $\beta \to 0$, *and actually we may take*

$$\alpha(\beta) = \frac{\beta}{2}\left(1 + \frac{\beta}{2}\right). \tag{5.22}$$

Thus for sufficiently small β *we have the implication*

$$(c, M) \in \mathcal{D}_\beta \Rightarrow (c-1, M') \in \mathcal{D}_\beta. \tag{5.23}$$

We will next establish (5.1) with

$$\gamma' = \gamma_{c,M} \tag{5.24}$$

(recall (5.7)), assuming the validity of (5.1) with c being replaced by $c-1$.

Proposition 5.4 *Let* z_0 *be the maximizer in the maximization over* z *of* (5.12), *and let* a *be associated with* z_0 *as in* (4.11), (4.13) *and* (4.17). *Let* γ' *be as in* (5.24) *and* $\gamma = \gamma_0$ *(recall* (5.10)*), and assume that* $(c, M) \in \mathcal{D}_{1/2}$. *Then* (5.4) *holds for some* $\alpha > 0$.

Proof: We consider the expression

$$f(\gamma) = f(\gamma_0) = \ln\left(1 + e^{\alpha}\frac{\ln M}{c-1}\right). \tag{5.25}$$

Let $K > 0$ be such that

$$\ln(1 + Kx) > 2x \text{ for every } 0 < x < 1/2$$

(e.g. $K = 4$). We then have that for some $\alpha_0 > 0$

$$\ln\left(1 + e^{\alpha}\frac{\ln M}{c-1}\right) > 2\frac{\ln M}{c} \tag{5.26}$$

for every $\alpha > \alpha_0$, if $(c, M) \in \mathcal{D}_{1/2}$. It then follows from (5.25) that

$$f(\gamma) > 2\frac{\ln M}{c},$$

and to establish (5.5) it is enough to verify

$$\frac{\gamma}{c} - \frac{\gamma'}{c+1} > -\frac{\ln M}{c}. \tag{5.27}$$

In view of

$$\gamma = a - \alpha + \ln(c-1) - \ln\ln M$$

and

$$\gamma' = a - \alpha + \ln c - \ln\ln M$$

we have

$$\frac{\gamma}{c} - \frac{\gamma'}{c+1} = \frac{a - \alpha - \ln\ln M}{c(c+1)} + \frac{\ln(c-1)}{c} - \frac{\ln c}{c+1},$$

and (5.27) follows from

$$\frac{\ln(c-1)}{c} - \frac{\ln c}{c+1} > -\frac{1}{c(c-1)}.$$

This concludes the proof of the proposition. $\square$

Remark 5.5 *Suppose that rather than* $(c, M) \in \mathcal{D}_{1/2}$ *we consider* $(c, M) \in \mathcal{D}_\beta$ *with* $\beta << 1$. *Arguing as in the above proof and analogous to* (5.26) *we consider the inequality*

$$\ln(1 + e^\alpha \beta) > (1 + \epsilon)\beta \tag{5.28}$$

for arbitrarily small $\epsilon > 0$. *For* $\epsilon = 0$ *let* (5.28) *hold for* $\alpha > \alpha(\beta)$, *and it is easy to see that we may take* $\alpha(\beta)$ *as in* (5.22).

It follows from (4.30), (5.4) and Proposition 5.4 that the inequality

$$F(c, M) > B \exp\left\{ M \left(1 - \frac{\ln M}{c+1} + \frac{a - \alpha + \ln c - \ln\ln M}{c+1} \right) \right\} \tag{5.29}$$

holds for certain values of α and certain pairs (c, M). Actually, the above discussion yields the next iterative property.

Proposition 5.6 *There exists an* $\alpha_0 > 0$ *such that for any fixed* $\alpha > \alpha_0$ *there exists* $\kappa_0 > 0$ *with the following property: If* $\kappa > \kappa_0$ *is such that* (5.29) *holds for every* $(c, M) \in \mathcal{D}_{1/2}$ *satisfying* $\kappa_0 < c \leq \kappa$, *then* (5.29) *also holds for every* (c, M) *that verifies*

$$(c, M) \in \mathcal{D}_{1/2} \text{ and } \kappa_0 < c \leq \kappa + 1.$$

To start the iterations procedure we need the following result:

Proposition 5.7 *For a fixed* $\alpha > \alpha_0$ *let* κ_0 *be as in Proposition* 5.6, *and let* B *be defined by*

$$B = e^{-\kappa_0 e^{a+\kappa_0}}. \tag{5.30}$$

Then (5.29) *holds for every* $(c, M) \in \mathcal{D}_{1/2}$ *such that* $c \geq \kappa_0$.

Proof: The assertion of the proposition follows from Propositions 4.4 and 5.6, employing an induction argument. □

We conclude from Propositions 5.6 and 5.7 the following result.

Proposition 5.8 *Let* $a < a^\star$ *be fixed. Then there exist* c_0, α *and* B *such that*

$$F(c, M) > B \exp\left\{ M \left(1 - \frac{\ln M + \ln\ln M}{c+1} + \frac{a - \alpha + \ln c}{c+1} \right) \right\} \tag{5.31}$$

for every (c, M) *such that* $M < e^{c/2}$ *and* $c > c_0$.

We next consider the expression

$$\bar{\gamma}_{c,M} = a - \alpha + \ln c + \ln\ln c - \ln\ln M - \ln\ln\ln M \tag{5.32}$$

instead of the expression $\gamma_{c,M}$ in (5.7), and repeat the above argument and computation using $\bar{\gamma}_{c,M}$ rather than $\gamma_{c,M}$. We will next indicate the required modifications.

Instead of (5.10) we have now

$$\gamma_0 = a - \alpha + \ln(c-1) + \ln^{(2)}(c-1) - \ln^{(2)} M - \ln^{(3)} M. \tag{5.33}$$

Proposition 5.2 and its proof still hold, where instead of (5.16) we have now

$$e^{\gamma_0 - a} = e^{-\alpha} \frac{c-1}{\ln M} \frac{\ln(c-1)}{\ln\ln M}. \tag{5.34}$$

We note that if (5.18) holds then $\ln c < \ln\ln M + \ln(1/\mu)$, implying that

$$\frac{\ln(c-1)}{\ln\ln M} < \frac{3}{2} \text{ if } c > c_0,$$

for some $c_0 > 0$. The rest of the proof of Proposition 5.2 applies in the present case without change.

Concerning the proof of Proposition 5.4, using the expression (5.33) for γ_0, we obtain

$$f(\gamma_0) = \ln\left(1 + e^{\alpha} \frac{\ln M}{c-1} \frac{\ln\ln M}{\ln(c-1)}\right). \tag{5.35}$$

We note that by $\ln M < c/2$ we have $\ln^{(2)} M / \ln(c-1) < 1$. Moreover, assuming that

$$c^{1-\theta} < \ln M \tag{5.36}$$

for some $0 < \theta < 1$ we obtain

$$\frac{\ln\ln M}{\ln c} > 1 - \theta. \tag{5.37}$$

Using (5.37) in (5.35) we can employ the rest of the proof of Proposition 5.4 to establish the following result.

Proposition 5.9 *Let z_0 be the maximizer in the maximization over z of* (5.12)*, and let a be associated with z_0 as in* (4.11)*,* (4.13) *and* (4.17)*. Let*

$$\gamma = a - \alpha + \ln(c-1) + \ln\ln(c-1) - \ln\ln M - \ln\ln\ln M \tag{5.38}$$

and

$$\gamma' = a - \alpha + \ln c + \ln\ln c - \ln\ln M - \ln\ln\ln M, \tag{5.39}$$

and assume that (5.36) *and $(c, M) \in \mathcal{D}_{1/2}$ are satisfied. Then there exists an α such that* (5.4) *holds.*

The following is the lower bound which we obtain for $F(c, M)$.

Theorem 5.10 *Consider pairs (c, M) such that*

$$c^{1-\theta} < \ln M < \frac{1}{2}c \tag{5.40}$$

for some $0 < \theta < 1$, and let $a < a^\star$ be fixed. Then there exist constants $\alpha > 0$ and $c_0 > 1$ such that

$$F(c,M) > \exp\left\{M\left(1 - \frac{\ln M + \ln^{(2)} M + \ln^{(3)} M}{c+1} + \frac{a - \alpha + \ln c + \ln^{(2)} c}{c+1}\right)\right\} \tag{5.41}$$

for every $c > c_0$.

Remark 5.11 *In view of Remarks* 5.3 *and* 5.5 *the discussion and proof which yield Theorem* 5.10 *can be employed to conclude the following: for any $\alpha > 0$, which may be arbitrarily small, we can choose $\beta > 0$ and $\theta > 0$ sufficiently small such that* (5.41) *holds for pairs (c, M) satisfying*

$$c^{1-\theta} < \ln M < \beta c, \tag{5.42}$$

replacing (5.40)*. Actually, in view of* (5.22)*, we may take $\beta < (2-\epsilon)\alpha$, if α is small enough.*

6 An upper bound for Problem $Q_{c,M}$

In this section we are concerned with the upper bound for $G(c, M)$ in (3.8). We will employ a method similar to the one used to establish a lower bound for $F(c, M)$ in sections 4 and 5.

It will be shown that the variables $G(c, M)$ satisfy relations similar to (3.9), and we wish to establish for $G(c, M)$ an inequality analogous to (3.10), with a reversed inequality sign. We note, however, that for fixed c, B and γ the inequality

$$G(c, M) \leq Be^{M\left(1 - \frac{\ln M}{c+1} + \frac{\gamma}{c+1}\right)} \tag{6.1}$$

cannot hold for sufficiently large M, since for such M the right-hand side of (6.1) becomes smaller than 1, while the left-hand side of (6.1) is clearly larger than 1.

We henceforth focus on the function $G(c, M)$ defined in (3.8). Our goal is to estimate the value of $G(c, M)$ for pairs (c, M) which belong to the domain

$$\mathcal{D} = \mathcal{D}_\beta \tag{6.2}$$

for some $0 < \beta < 1/2$ (recall (5.6)). We denote

$$\mathcal{D}_+ = \{(c, M) : e^{\beta c} < M < e^{\beta(c+1)}\} \tag{6.3}$$

and

$$\epsilon = 1 - 2\beta. \tag{6.4}$$

Analogous to (3.9), for points $(c, M) \in \mathcal{D}$ we have the following relation

$$G(c, M) = \sum_{z=0}^{[M/c]} G(c-1, M-cz) \frac{m_0^z}{z!} e^{z^2/m_0}. \tag{6.5}$$

(Of course, even though $(c, M) \in \mathcal{D}$, some points $(c-1, M-cz)$ in (6.5) may fail to belong to $\mathcal{D}$.)

To obtain an upper bound of the type (6.1) on $\mathcal{D}$ we will employ the iterative method described in sections 4 and 5. To use this approach in the present situation we have to guarantee in advance that (6.1) holds for points in $\mathcal{D}_+$. This property will be a consequence of the following results.

Proposition 6.1 *Let $\{p_k\}_{k=1}^\infty$ denote the sequence of primes. Then*

$$\psi(x, p_{k+1}) = \sum_{j=0}^{N_{k+1}} \psi\left(x/p_{k+1}^j, p_k\right) \tag{6.6}$$

holds for every $x > 2$ and $k \geq 1$, where we denote $N_k = \left[\dfrac{\ln x}{\ln p_{k+1}}\right]$.

Proof: Let $\mathcal{F}_k(x)$ denote the set of integers $z \leq x$ whose largest prime divisor does not exceed p_k, so that

$$\psi(x, p_k) = \#\{\mathcal{F}_k(x)\}. \tag{6.7}$$

Denote by A_j the set of integers $z \in \mathcal{F}_{k+1}(x)$ such that p_{k+1}^j is the largest power of p_{k+1} which divides z. It is then easy to see that

$$A_j = p_{k+1}^j \mathcal{F}_k\left(\frac{x}{p_{k+1}^j}\right) \tag{6.8}$$

and

$$\mathcal{F}_{k+1}(x) = \bigcup_{j \geq 0} A_j, \tag{6.9}$$

a disjoint union. Equation (6.6) follows from (6.7), (6.8) and (6.9). □

Proposition 6.2 *Let $\alpha > 1$ be fixed, and consider pairs (x, y) such that*

$$y = \alpha(\ln x)^2. \tag{6.10}$$

Then there exists a constant $C > 1$ such that

$$\frac{\ln \psi(x, y)}{\ln x} < \frac{1}{2} + \frac{C}{\ln y} \tag{6.11}$$

holds for every $x > 1$, where y is as in (6.10).

The proof is displayed in the appendix.

Proposition 6.3 *Let $\mathcal{D}$ be as in* (6.2). *Then there exist constants K and c_0 such that*

$$G(c, M) < KF(c, M) \tag{6.12}$$

holds for every $c \geq c_0$.

Proof: We note that

$$z \leq \frac{M}{c} \leq \frac{e^{\beta(c+1)}}{c} \text{ and } m_0 > \frac{e^c}{c},$$

implying

$$\frac{z^2}{m_0} < \frac{e}{ce^{\epsilon c}}, \tag{6.13}$$

and it follows that

$$e^{z^2/m_0} < \exp\{3e^{-\epsilon c}/c\}.$$

We fix a constant c_0, and then (6.12) follows from (3.9), (6.5) and (6.13) for $c \geq c_0$, by employing induction on c. $\square$

Remark 6.4 *We will establish an upper bound for $F(c, M)$, and then use* (6.12) *to estimate $G(c, M)$ from above. Thus we wish to establish for F an inequality of the form*

$$F(c, M) \leq B_1 e^{M\left(1 - \frac{\ln M}{c+1} + \frac{\gamma}{c+1}\right)} \tag{6.14}$$

for some coefficient B and a certain γ (which may depend on c and M), and in view of (6.12) *this will yield the estimate*

$$G(c, M) \leq B \exp\left\{M\left(1 - \frac{\ln M}{c+1} + \frac{\gamma}{c+1}\right)\right\}. \tag{6.15}$$

The following result is a consequence of Proposition 6.2.

Proposition 6.5 *Let $\mathcal{D}_+$ be as in* (6.3)*, and let C be as in Proposition* 6.2*. Then* (6.14)*, with $B = 1$ and $\gamma = C$, holds on $\mathcal{D}_+$.*

We consider (6.5) as a difference equation in $\mathcal{D}$ satisfying boundary upper bounds on $\mathcal{D}_+$ as expressed in Proposition 6.5. For a fixed $\kappa > 1$ let

$$E_\kappa = \mathcal{D} \cap \{1 \leq c \leq \kappa\}$$

which is a bounded set, and it follows that for any fixed γ, $F(\cdot, \cdot)$ satisfies (6.14) on E_κ for some $B > 1$ (depending on γ).

Suppose that we have an upper bound for $F(\cdot, \cdot)$ on E_κ, and we consider in the left hand side of (3.9) pairs (c, M) which belong to $E_{\kappa+1} \setminus E_\kappa$. We will next show that for such (c, M) the right hand side of (3.9) involves pairs $(c-1, M-cz)$ for which an upper bound of the form (6.14) has been already established. We will then use these bounds to estimate the right hand side of (3.9), thus establishing an upper bound for $F(c, M)$.

Proposition 6.6 *If* $(c, M) \in E_{\kappa+1} \setminus E_\kappa$ *then*

$$(c-1, M-cz) \in E_\kappa \cup \mathcal{D}_+ \tag{6.16}$$

for every $0 \leq z \leq M/c$.

Proof: If $(c, M) \in E_{\kappa+1}$ then $M \leq e^{\beta c}$. Obviously this can be written in the form

$$M \leq e^{\beta[(c-1)+1]},$$

implying that $(c-1, M) \in \mathcal{D}_+$ if $M > e^{\beta(c-1)}$, and $(c-1, M) \in E_\kappa$ if $M \leq e^{\beta(c-1)}$. □

It follows from Proposition 6.6 that each summand $F(c-1, M-cz)$ in the right hand side of (6.5) may be bounded by employing a bound of the form (6.14) for $(c-1, M-cz)$.

In analogy with (4.6) we have that

$$\frac{m_0^z}{z!} < e^{\bar{E}}, \tag{6.17}$$

where similarly to (4.14)

$$\bar{E} = (z \ln m_0 - z \ln z + z). \tag{6.18}$$

(In (6.18) we ignore the term $\sqrt{z}$ in (4.8), since we consider now an upper bound.) Substituting $m_0 = (e-1)e^c/c$ in (6.18) we obtain

$$\bar{E} = cz - z \ln z + z(1 + \ln(e-1)) - z \ln c.$$

Let A be as in (4.5), and analogous to (4.4) we assume that

$$F(c-1, M-cz) \leq Be^A,$$

so that

$$F(c-1, M-cz)\frac{m_0^z}{z!} \leq Be^{A+\bar{E}}.$$

It follows that an upper bound for $A + \bar{E}$ is given by the function $H(z)$ in (4.16), where the variable a (recall (4.17)) is replaced by $a^\star$ in (4.18). We still denote this function by $H(z)$, and analogous to (4.19) we have the relation

$$F(c-1, M-cz)\frac{m_0^z}{z!} < Be^{H(z)}. \tag{6.19}$$

As in section 4, we should maximize the function $H(z)$ over $0 \le z \le [M/c]$. But in the present situation, since we are concerned with an upper bound, we may use the maximum of $H(z)$ over the real interval $0 \le z \le M/c$ and do not have to restrict to the integers in this interval.

Summarizing the above discussion we obtain, analogous to (4.30), the following result.

Proposition 6.7 *Assume that*

$$F(c, M) \le Be^{M\left(1-\frac{\ln M}{c+1}+\frac{\gamma}{c+1}\right)} \tag{6.20}$$

for every $(c, M) \in E_\kappa$*, for some* $\gamma > C$ *and* $\kappa > 1$*. Then*

$$\max\left\{F(c-1, M-cz)\frac{m_0^z}{z!} : 0 \le z \le \frac{M}{c}\right\} \le Be^{M\left(1-\frac{\ln M}{c}+\frac{\gamma+f(\gamma)}{c}\right)}, \tag{6.21}$$

implying

$$F(c, M) < Be^{M\left(1-\frac{\ln M}{c}+\frac{\gamma+f(\gamma)}{c}\right)+\ln(M/c)} \tag{6.22}$$

for every $(c, M) \in E_{\kappa+1}$*.*

Remark 6.8 *The term* $\ln(M/c)$ *appears in* (6.22) *since we should multiply the maximum in* (6.21) *by the number of terms which appear in the sum in* (3.9)*. We may use* $\ln(M/c)$ *rather than* $\ln([M/c]+1)$ *since there are in* (3.9) *several summands which are much smaller than the maximal term there.*

In this section we use induction to establish an inequality of the type (6.14), with γ depending on (c, M) as follows:

$$\gamma(c, M) = \bar{a} + \ln c + \ln\ln c - \ln^{(2)} M - \ln^{(3)} M \tag{6.23}$$

for a certain $\bar{a} > a^\star$.

We consider now the maximization in the left hand side of (6.21). Employing an induction hypothesis we obtain bounds on the expressions $F(c-1, M-cz)$, using inequalities of the form (6.20) for the pairs $(c-1, M')$, where $M' = M - cz$. In these bounds we denote $\gamma = \gamma(c-1, M')$, using (6.23). Suppose that the *maximum over the bounds* is attained at $1 < M_0 \le M$, and denote $\gamma_0 = \gamma(c-1, M_0)$, namely

$$\gamma_0 = \bar{a} + \ln(c-1) + \ln\ln(c-1) - \ln^{(2)} M_0 - \ln^{(3)} M_0. \tag{6.24}$$

Clearly the maximum over the bounds is not larger than the maximal value of

$$\exp\left\{(M-cz)\left[1-\frac{\ln(M-cz)}{c}+\frac{\gamma_0}{c}\right]\right\}\frac{m_0^z}{z!} \tag{6.25}$$

over $0 \le z \le M/c$.

In view of (6.21) and (6.22), and analogous to (5.5), we wish to establish

$$\frac{\gamma_0 + f(\gamma_0)}{c} < \frac{\ln M}{c(c+1)} + \frac{\gamma'}{c+1}, \tag{6.26}$$

where

$$\gamma' = \bar{a} + \ln c + \ln\ln c - \ln^{(2)} M - \ln^{(3)} M. \tag{6.27}$$

We first address the term $f(\gamma_0)$ in (6.26), and recalling (4.24) and (6.24) we have

$$f(\gamma_0) = \ln\left(1 + e^{a^\star - \bar{a}}\frac{\ln M_0 \ln^{(2)} M_0}{(c-1)\ln(c-1)}\right). \tag{6.28}$$

We assume now that

$$(c, M) \in \mathcal{D}_\beta,$$

and denote in (6.23)

$$\bar{a} = a^\star + \delta \tag{6.29}$$

for some $\delta > 0$. For small enough β, arguing as in Remark 5.3 we have, analogous to (5.21)

$$\frac{\ln M_0}{c-1} < (1+\epsilon)\beta.$$

Clearly we have also $\frac{\ln\ln M_0}{\ln(c-1)} < 1$, and thus, if δ is sufficiently small, then

$$\frac{f(\gamma_0)}{c} < \frac{q\ln M}{c(c+1)} \text{ if } c > c_0 \tag{6.30}$$

for some c_0, where

$$e^{-\delta} < q < 1. \tag{6.31}$$

We note that q in (6.31) may be arbitrarily close to $e^{-\delta}$ provided that $\beta > 0$ and $\delta > 0$ are sufficiently small. Specifically we may choose the parameters δ and q in (6.29), (6.30) and (6.31) as follows:

$$\delta = \lambda\beta \text{ and } q = 1 - \frac{\lambda\beta}{2} \tag{6.32}$$

where $\lambda > 0$ may be arbitrarily small.

We next consider the terms γ_0/c and $\gamma'/(c+1)$ in (6.26). Let z_0 be the point where the maximization over z of (6.25) is attained, and let, as above, $M_0 = M - cz_0$. We note that in this maximization, the value γ_0 is the same for all the points $(c-1, M')$, $1 < M' \le M$. We have then

$$M_0 = M(1 - t_0), \tag{6.33}$$

where by (4.27)

$$t_0 = \frac{1}{1 + e^{\gamma_0 - a^\star}} < \frac{e^{-\delta} \ln M_0 \ln^{(2)} M_0}{c \ln c}.$$

Thus

$$\ln(1 - t_0) > -\frac{q_1 \ln M_0 \ln^{(2)} M_0}{c \ln c} \tag{6.34}$$

for some constant $e^{-\delta} < q_1 < 1$. It follows from (6.33) and (6.34) that

$$\left(1 + \frac{q_1 \ln^{(2)} M_0}{c \ln c}\right) \ln M_0 > \ln M,$$

hence

$$\ln M_0 > \left(1 - \frac{q_1 \ln^{(2)} M_0}{c \ln c}\right) \ln M,$$

and we obtain

$$\ln \ln M_0 > \ln \ln M - \frac{q_1 \ln^{(2)} M}{c \ln c} \tag{6.35}$$

for some constant q_2.

Using the expressions (6.24) and (6.27) it follows from (6.35) that $\dfrac{\gamma_0}{c} - \dfrac{\gamma'}{c+1}$ is smaller than

$$\frac{\bar{a} + \ln(c-1) + \ln \ln(c-1)}{c} - \frac{\bar{a} + \ln c + \ln \ln c}{c+1} + \frac{q_2 \ln^{(2)} M}{c^2 \ln c},$$

implying that

$$\frac{\gamma_0}{c} - \frac{\gamma'}{c+1} < \frac{\bar{a} + \ln c + \ln \ln c}{c(c+1)} + \frac{q_2 \ln^{(2)} M}{c^2 \ln c} \tag{6.36}$$

If (c, M) is such that

$$\ln c < (1-q)\ln M,$$

then (6.26) would follow from (6.30) and (6.36) for large enough c. We thus consider pairs (c, M) satisfying

$$M > c^\nu \tag{6.37}$$

for some constant $\nu > 1$ such that

$$(1-q)\nu \geq 1. \tag{6.38}$$

If we choose, as in (6.32), $q = 1 - \lambda\beta/2$ for some $\lambda > 0$, we may take

$$\nu = \frac{2}{\lambda\beta}. \tag{6.39}$$

We have thus established the following result.

Proposition 6.9 *Let $\bar{a}$ and $\delta > 0$ be as in* (6.29)*, let $\gamma(c, M)$ be as in* (6.23)*, and consider pairs $(c, M) \in \mathcal{D}$ which satisfy* (6.37) *and* (6.38)*. Then there exist constants B, c_0 and δ_0 such that*

$$F(c, M) < Be^{M\left(1-\frac{\ln M}{c+1}+\frac{\gamma(c,M)}{c+1}\right)+c\ln(M/c)} \tag{6.40}$$

holds provided that $c > c_0$ and $\delta > \delta_0$.

Proof: The inequality (6.40) follows from (6.22) and (6.26) and the preceding discussion. We note that when employing successively the inequalities (6.22) and (6.26), the various terms $\ln(M/c)$ in (6.22) accumulate, yielding the term $c\ln M$ in (6.40). □

Concerning $G(c, M)$, in view of Remark 6.4 we obtain the following result:

Theorem 6.10 *Consider pairs (c, M) satisfying*

$$c^\nu < M < e^{\beta c} \tag{6.41}$$

for some $\nu > 2$ and $0 < \beta < 1/2$. Then there exist constants $\overline{a} > a^\star$ and c_0 such that

$$G(c, M) < e^{M\left(1-\frac{\ln M+\ln^{(2)} M+\ln^{(3)} M}{c}+\frac{\overline{a}+\ln c+\ln^{(2)} c}{c}\right)} \tag{6.42}$$

for every $c > c_0$. Moreover, for every $\lambda > 0$, which may be arbitrarily small, we may take

$$\overline{a} = a^\star + \lambda\beta$$

provided that $\beta > 0$ is sufficiently small and $\nu \geq 2/\lambda\beta$.

The last assertion of the theorem follows from (6.39).

7 The main results

In this section we will establish our main results concerning lower and upper bounds for $\psi(x,y)$. They consist of rephrasing the results in sections 5 and 6 in terms of x and y instead of c and M. We obtain from Theorem 5.10 our first main result:

Theorem 7.1 *Consider* (x,y) *such that*

$$\exp\{(\ln y)^{1-\theta}\} < \ln x < \sqrt{y} \tag{7.1}$$

for some $\theta > 0$. *Then there exists an* $\underline{a}$ *and a* y_0 *such that*

$$\frac{\ln \psi(x,y)}{\ln x} > 1 - \frac{\ln^{(2)} x + \ln^{(3)} x + \ln^{(4)} x}{\ln y} + \frac{\underline{a} + \ln^{(2)} y + \ln^{(3)} y}{\ln y} \tag{7.2}$$

for every (x,y) *satisfying* (7.1) *and* $y > y_0$.

Concerning an upper bound for $\psi(x,y)$, Theorem 6.10 yields our second main result:

Theorem 7.2 *For some constants* $0 < \beta < 1/2$ *and* $\nu > 0$ *consider pairs* (x,y) *which satisfy*

$$(\ln y)^{\nu} < \ln x < y^{\beta}, \tag{7.3}$$

and let u *be as in* (1.8). *Then there exist constants* y_0 *and* $\overline{a} > a^{\star}$ *such that*

$$\frac{\ln \psi(x,y)}{\ln x} < 1 - \frac{\ln^{(2)} x + \ln^{(3)} x + \ln^{(4)} x}{\ln y} + \frac{\overline{a} + \ln^{(2)} y + \ln^{(3)} y}{\ln y} + \frac{\ln u}{u} \tag{7.4}$$

holds provided that $y > y_0$. *If in* (7.3) *we have* $\nu > 2$ *then*

$$\frac{\ln \psi(x,y)}{\ln x} < 1 - \frac{\ln^{(2)} x + \ln^{(3)} x + \ln^{(4)} x}{\ln y} + \frac{\overline{a} + \ln^{(2)} y + \ln^{(3)} y}{\ln y} \tag{7.5}$$

holds for every $y > y_1$, *for some* y_1. *Moreover, we may take* $\overline{a} > a^{\star}$ *to be arbitrarily close to* $a^{\star}$ *provided that* β *is small enough and* ν *is large enough.*

Remark 7.3 *The bounds* (7.2) *and* (7.5) *raise the conjecture that for each* $k \geq 2$*, in a certain range of the variables* x *and* y *the following bounds*

$$\frac{\ln \psi(x,y)}{\ln x} > 1 - \frac{1}{\ln y}\left[\sum_{j=2}^{k+1} \ln^{(j)} x - \underline{a} - \sum_{j=2}^{k} \ln^{(j)} y\right] \tag{7.6}$$

and

$$\frac{\ln \psi(x,y)}{\ln x} < 1 - \frac{1}{\ln y}\left[\sum_{j=2}^{k+1} \ln^{(j)} x - \overline{a} - \sum_{j=2}^{k} \ln^{(j)} y\right] \tag{7.7}$$

are valid for certain constants $\underline{a}$ *and* $\overline{a}$*.*

Remark 7.4 *The inequalities* (7.2) *and* (7.5) *may be written in the form*

$$\ln\left(\frac{\psi(x,y)}{x}\right) > -u[\ln u + \ln^{(3)} x - \ln^{(3)} y + \ln^{(4)} x - \underline{a}] \tag{7.8}$$

and

$$\ln\left(\frac{\psi(x,y)}{x}\right) < -u[\ln u + \ln^{(3)} x - \ln^{(3)} y + \ln^{(4)} x - \overline{a}] \tag{7.9}$$

respectively, where we used

$$\ln^{(2)} x - \ln^{(2)} y = \ln u. \tag{7.10}$$

We will next estimate the value of iterated logarithms $\ln^{(k)} x$ and $\ln^{(k)} y$ for pairs (x, y) which satisfy

$$\exp(\ln y)^{\nu} < \ln x < y^{\beta} \tag{7.11}$$

for some $0 < \nu < 1$. To do this we will use the iterated logarithms $\ln^{(k)} u$.

Proposition 7.5 *Let* (x, y) *be such that* (7.11) *holds, and let* u *be as in* (1.8)*. Then*

(i) For every $k \geq 3$

$$\ln^{(k)} x = \ln^{(k-1)} u + o(1). \tag{7.12}$$

(ii) For every $k \geq 4$

$$\ln^{(k)} y = \ln^{(k)} u + o(1), \tag{7.13}$$

and

$$\ln^{(3)} u + o(1) < \ln^{(3)} y < \ln^{(3)} u + \ln(1/\nu) + o(1). \tag{7.14}$$

Proof: It follows from the left inequality in (7.11) that

$$\ln^{(2)} y < \frac{1}{\nu} \ln^{(3)} x. \tag{7.15}$$

We conclude from (7.10) and (7.15) that

$$\ln u < \ln^{(2)} x < \ln u + \frac{1}{\nu} \ln^{(3)} x. \tag{7.16}$$

Since

$$\frac{\ln^{(3)} x}{\ln^{(2)} x} = o(1) \text{ as } x \to \infty,$$

it follows from (7.16) that

$$\ln^{(2)} x = (\ln u)(1 + o(1)),$$

which establishes (7.12) for every $k \geq 3$.

Concerning $\ln^{(3)} y$ we obtain from the right inequality in (7.11) that

$$\beta \ln y > \ln u + \ln^{(2)} y. \tag{7.17}$$

Since

$$\frac{\ln^{(2)} y}{\ln y} = o(1) \text{ as } y \to \infty,$$

we conclude from (7.17) that

$$\beta(1 + o(1)) \ln y > \ln u,$$

and therefore

$$\ln^{(2)} y > \ln^{(2)} u - \ln \beta + o(1),$$

implying

$$\ln^{(3)} y > \ln^{(3)} u + o(1). \tag{7.18}$$

Moreover, it follows from from the left inequality in (7.11) that

$$\ln u + \ln^{(2)} y > (\ln y)^{\nu}. \tag{7.19}$$

Since

$$\frac{\ln^{(2)} y}{(\ln y)^{\nu}} = o(1)$$

we obtain

$$(\ln y)^\nu < (1 + o(1)) \ln u.$$

This implies

$$\ln^{(3)} y < \ln^{(3)} u - \ln \nu + o(1),$$

which together with (7.18) establishes (7.14). The relations (7.13) for $k \geq 4$ follow from (7.14). The proof of the proposition is complete. $\square$

The estimates in Proposition 7.5 yield the approximation of $\ln \rho(u)$ presented in Corollary 1.2.

We conclude this section by employing Theorems 7.1 and 7.2 to establish a result concerning Bertrand's Conjecture. As is well known, Bertrand's conjecture was that for every integer y there exists a prime p satisfying $y \leq p \leq 2y$.

Corollary 7.6 *Let $\gamma > 3/2$ be fixed. Then there exists a y_0 such that for every integer $y > y_0$ there exists a prime p satisfying*

$$y < p < \gamma y. \tag{7.20}$$

Proof: By Remark 5.5 and Theorem 6.10 we may assume that

$$|a^\star - \underline{a}| < \frac{\beta}{2} \text{ and } |a^\star - \overline{a}| < \left(\gamma - \frac{3}{2}\right) \beta$$

provided that β is small enough, and that ν and $\ln x / \ln y$ are large enough. We thus assume that the latter parameters were chosen such that

$$|\overline{a} - \underline{a}| < (\gamma - 1)\beta, \tag{7.21}$$

and such that there exists a y_0 for which

$$\exp\{(\ln y)^\nu\} < \ln x < y^\beta \tag{7.22}$$

holds for $y = y_0$ some x_0. Then (7.22) holds for every $y > y_0$, and we may assume that y_0 and x_0 were chosen such that $\ln x_0 / \ln y_0$ is sufficiently large, as required. For $y_1 > y_0$ denote $y_2 = \gamma y_1$, and let x be such that both (x, y_1) and (x, y_2) satisfy (7.22).

We write the inequalities (1.9) and (1.11) in the form

$$\ln \psi(x, y) = u[\ln y + \ln^{(2)} y + \ln^{(3)} y - \ln^{(2)} x - \ln^{(3)} x - \ln^{(4)} x + a] \tag{7.23}$$

where a satisfies $\underline{a} < a < \overline{a}$, and employ (7.23) to estimate $\psi(x, y_2) - \psi(x, y_1)$. For a fixed value of a we denote by $\psi_a(x, y)$ the expression for $\psi(x, y)$ in (7.23). To estimate $\psi_a(x, y_2) - \psi_a(x, y_1)$ we consider the partial derivative $(\ln \psi_a(x, y))_y$, which is equal to

$$\ln x \frac{\partial}{\partial y} \left(\frac{1}{\ln y} [\ln y + \ln^{(2)} y + \ln^{(3)} y - \ln^{(2)} x - \ln^{(3)} x - \ln^{(4)} x + a] \right). \quad (7.24)$$

It is easy to see that the expression (7.24) is larger than

$$\frac{\ln x \ln u}{y_1 (\ln y_1)^2} \text{ for every } y_1 \leq y \leq y_2. \quad (7.25)$$

The fact that $(\ln \psi_a(x, y))_y$ is larger than the expression in (7.25) implies that

$$\psi_a(x, y_2) > \psi_a(x, y_1) \exp \left\{ \frac{(\gamma - 1) u \ln u}{\ln y_1} \right\}, \quad (7.26)$$

where we used $y_2 - y_1 = (\gamma - 1) y_1$

Returning to (7.23) let a_1 and a_2 correspond to y_1 and y_2 in this formula, so that by (7.21)

$$|a_2 - a_1| < |\overline{a} - \underline{a}| < (\gamma - 1)\beta,$$

and we write

$$|a_2 - a_1| = \sigma\beta, \ \sigma < \gamma - 1. \quad (7.27)$$

It follows from (7.23), (7.28) and (7.27) that

$$\psi(x, y_2) > \psi(x, y_1) \exp \left\{ \frac{(\gamma - 1) u \ln u}{\ln y_1} - \sigma\beta u \right\}. \quad (7.28)$$

By (7.22) the pair (x, y_1) satisfies

$$\frac{\ln \ln x}{\ln y_1} < \beta,$$

and moreover, taking x sufficiently large we can have $\ln^{(2)} x / \ln y_1$ be arbitrarily close to β. In this case we also have

$$\left| 1 - \frac{1}{\beta} \frac{\ln u}{\ln y_1} \right| \text{ is arbitrarily small} \quad (7.29)$$

provided that x is sufficiently large. Writing the exponent in the right hand side of (7.28) in the form

$$(\gamma-1)\beta u\left[\frac{1}{\beta}\frac{\ln u}{\ln y_1}-\frac{\sigma}{\gamma-1}\right] \tag{7.30}$$

yields, in view of (7.29) and $\sigma<\gamma-1$, that

$$\psi(x,y_2)>2\psi(x,y_1),$$

if x is large enough, from which we conclude that

$$\psi(x,y_2)-\psi(x,y_1)>2. \tag{7.31}$$

But clearly (7.31 implies that there exists a prime p satisfying $y_1<p<y_2$. This establishes (7.20), and completes the proof of the corollary. $\square$.

8 Appendix

Proof of Theorem 1.5: Let $F=[1,x]\setminus E$ be the complement of E in $[1,x]$. For a prime $\sqrt{x}\le p\le x$ we denote by F_p the set of integers in F which are divisible by p. Then $F_{p_1}\cap F_{p_2}=\emptyset$ if $p_1\ne p_2$,

$$\#(F_p)=\left[\frac{N}{p}\right]$$

and it follows that

$$\#(F)=\sum_{\sqrt{x}\le p\le x}[x/p]<x\sum_{p\ge\sqrt{x}}^{x}\frac{1}{p}, \tag{8.1}$$

where the sum is over the primes in the indicated interval. To estimate the sum in the right hand side of (8.1) we consider, more generally, sums of the form

$$S_{a,b}=\sum_{a\le p\le b}\frac{1}{p}. \tag{8.2}$$

By the Prime Numbers Theorem the distribution function of the number of primes in the real line is, for large enough t, $\Phi(t) = t/\ln t$. Using this in the summation in (8.2) implies that for sufficiently large a we have

$$S_{a,b} \approx \int_a^b \frac{d\Phi(t)}{t} = \int_a^b \frac{\Phi(t)dt}{t^2} + \left.\frac{\Phi(t)}{t}\right|_a^b,$$

and substituting $\Phi(t) = t/\ln t$ we conclude that

$$S_{a,b} \approx \int_a^b \frac{dt}{t\ln t} + \left.\frac{1}{\ln t}\right|_a^b < \ln\ln b - \ln\ln a. \tag{8.3}$$

For $a = \sqrt{x}$ and $b = x$ the right hand side of (8.3) is equal to $\ln 2$, and using this in (8.1) yields that for sufficiently large x we have

$$\#(F) < x\ln 2,$$

implying

$$\#(E) > x\ln(e/2).$$

This establishes (1.18) and concludes the proof. $\square$.

Proof of Proposition 6.11: It follows from $\psi(x,2) \leq \ln x/\ln 2$ that

$$\psi(x,2) \leq \frac{\sqrt{x}}{\ln 2},$$

since $\ln x < \sqrt{x}$ for every $x \geq 1$. It is easy to see that

$$\psi(x,p_k) \leq \frac{\sqrt{x}}{(\ln 2)(1-1/\sqrt{p_2})\cdots(1-1/\sqrt{p_k})}, \tag{8.4}$$

for every $k \geq 2$. Relation (8.4) can be established by employing a simple induction argument, using (6.6).

To estimate from above the right hand side of (8.4), we have to estimate from below the product

$$\prod_{j=1}^{k}\left(1-\frac{1}{\sqrt{p_j}}\right), \tag{8.5}$$

and for this we estimate from above the sum

$$\sum_{j=1}^{k}\frac{1}{\sqrt{p_j}}. \tag{8.6}$$

To this end we use the distribution function

$$\Phi(t) = \frac{t}{\ln t}$$

of the primes in the real line, and we have to estimate

$$\int_3^{p_k} \frac{d\Phi(t)}{\sqrt{t}}.$$

This leads to

$$\int_3^{p_k} \frac{dt}{\sqrt{t}\ln t} = \int_{\sqrt{3}}^{\sqrt{p_k}} \frac{ds}{2\ln s} < \frac{C\sqrt{p_k}}{\ln p_k} \tag{8.7}$$

for some constant $C > 0$, and we obtain

$$\psi(x, p_k) \leq x^{1/2} \exp\left\{C\sqrt{p_k}/\ln p_k\right\}. \tag{8.8}$$

For a prescribed $y = \alpha(\ln x)^2$ we let p_k be the smallest prime p which satisfies $p \geq y$. Employing (8.8) for this p_k yields the assertion of the proposition. □

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